

# More on Combinatorial interpretation of the fibonomial coefficients

*Andrzej K. Kwaśniewski*

Higher School of Mathematics and Applied Informatics  
Kamienna 17, PL-15-021 Białystok , Poland

ArXiv : math.CO/0403017 v 1 1 March 2004

## Summary

Combinatorial interpretation of the fibonomial coefficients recently attempted by the present author [1,2] and presented here with suitable improvements results in a proposal of a might be combinatorial interpretation of the recurrence relation for fibonomial coefficients. The presentation is provided within the context of the classical combinatorics' attitude to that type of basic enumerative problems [3,4]. This note apart from plane grid coordinate system used is fitted with several figures (Fig.1 - corrected) and examples which illustrate the exposition of statements and an interpretation of the recurrence itself.

## 1 Introduction

There are various classical interpretations of binomial coefficients, Stirling numbers, and the  $q$ -Gaussian coefficients. Recently Kovanlina have discovered [3,4] a simple and natural unified combinatorial interpretation of all of them in terms of object selection from weighted boxes with and without box repetition. So we are now in a position of the following recognition:

The classical, historically established standard interpretations might be schematically presented for the sake of hint as follows:

SETS : Binomial coefficient  $\binom{n}{k}$ ,  $\binom{n+k-1}{k}$  denote number of subsets (without and with repetitions) - i.e. we are dealing with LATTICE of subsets.

SET PARTITIONS: Stirling numbers of the second kind  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  denote number of partitions into exactly  $k$  blocs - i.e. we are dealing with LATTICE of partitions.

PERMUTATION PARTITIONS : Stirling numbers of the first kind  $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right]$  denote number of permutations containing exactly  $k$  cycles

SPACES:  $q$ -Gaussian coefficient  $\binom{n}{k}_q$  denote number of  $k$ -dimensional subspaces in  $n$ -th dimensional space over Galois field  $GF(q)$  [5,6,7] i.e. we are dealing with LATTICE of subspaces.

### Before Konvalina combinatorial interpretation.

Algebraic similarities of the above classes of situations provided Rota [5] and Goldman and Rota [8,9] with an incentive to start the algebraic unification that captures the intrinsic properties of these numbers. The binomial coefficients, Stirling numbers and Gaussian coefficients appear then as the coefficients in the characteristic polynomials of geometrical lattices [5] (see also [10] for the subset-subspace analogy). The generalized coefficients [3] are called Whitney numbers of the first (characteristic polynomials) and the second kind (rank polynomials).

## Konvalina combinatorial interpretation

All these cases above and the case of Gaussian coefficients of the first kind  $q^{\binom{n}{2}} \binom{n}{k}_q$  are given unified Konvalina combinatorial interpretation in terms of the **generalized** binomial coefficients of the first and of the second kind (see: [3,4] ).

**Unknowns ?** As for the distinguished [11,12,13,14] Fibonomial coefficient defined below - no combinatorial interpretation was known till today now to the present author . The aim of this note is to promote a long time waited for - classical in the spirit - combinatorial interpretation of Fibonomial coefficients. Namely we propose following [1,2] such a partial ordered set that the Fibonomial coefficients count the number of specific finite “*birth – selfsimilar*” sub-posets of an infinite locally finite not of binomial type , non-tree poset naturally related to the Fibonacci tree of rabbits growth process. This partial ordered set is defined equivalently via  $\zeta$  characteristic matrix of partial order relation from its Hasse diagram. The classical scheme to be continued through ”Fibonomials” interpretation is the following:

POSET : Fibonomial coefficient  $\binom{n}{k}_F$  is the number of “birth-selfsimilar” sub-posets.

## 2 Combinatorial Interpretation

It pays to get used to write  $q$  or  $\psi$  extensions of binomial symbols in mnemonic convenient upside down notation [16,17] .

$$\begin{aligned} (1) \quad & \psi_n \equiv n_\psi, x_\psi \equiv \psi(x) \equiv \psi_x, n_\psi! = n_\psi(n-1)_\psi!, n > 0, \\ (2) \quad & x_\psi^k = x_\psi(x-1)_\psi(x-2)_\psi \dots (x-k+1)_\psi \\ (3) \quad & x_\psi(x-1)_\psi \dots (x-k+1)_\psi = \psi(x)\psi(x-1) \dots \psi(x-k-1). \end{aligned}$$

You may consult [16,17] for further development and profit from the use of this notation . So also here we use this upside down convention for Fibonomial coefficients:

$$\binom{n}{k}_F = \frac{F_n!}{F_k!F_{n-k}!} \equiv \frac{n_F^k}{k_F!}, \quad n_F \equiv F_n \neq 0,$$

where we make an analogy driven [16,17] identifications ( $n > 0$ ):

$$\begin{aligned} n_F! &\equiv n_F(n-1)_F(n-2)_F(n-3)_F \dots 2_F 1_F; \\ 0_F! &= 1; \quad n_F^k = n_F(n-1)_F \dots (n-k+1)_F. \end{aligned}$$

This is the specification of the notation from [16] for the purpose Fibonomial Calculus case (see Example 2.1 in [17]).

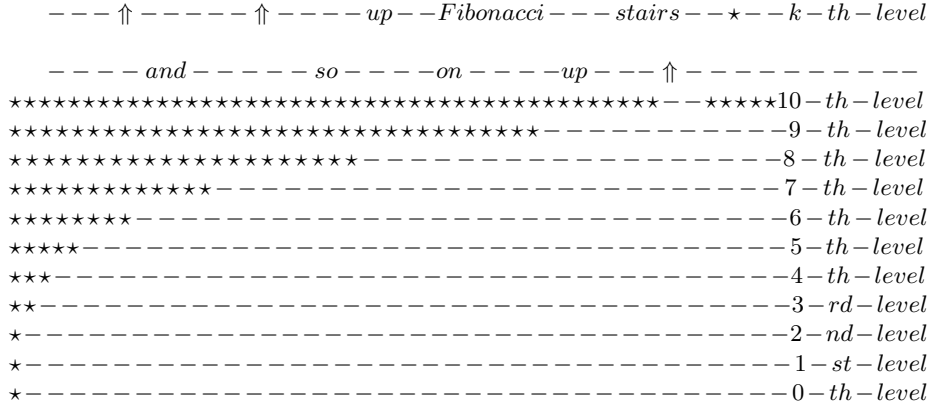
Let us now define the partially ordered infinite set  $P$ . We shall label its vertices by pairs of coordinates:  $\langle i, j \rangle \in N \times N_0$  where  $N_0$  denotes the nonnegative integers. Vertices show up in layers (“generations”) of  $N \times N_0$  grid along the recurrently emerging subsequent  $s$ -th levels  $\Phi_s$  where  $s \in N_0$  i.e.

### Definition 1

$$\Phi_s = \{\langle j, s \rangle | 1 \leq j \leq s_F\}, s \in N_0.$$

We shall refer to  $\Phi_s$  as to the set of vertices at the  $s$ -th level. The population of the  $k$ -th level (“generation”) counts  $k_F$  different member vertices for  $k > 0$  and one for  $k = 0$ .

Here down a disposal of vertices on  $\Phi_k$  levels is visualized.



**Figure 0.** The  $s$ -th levels in  $N \times N_0$ ,  $N_0$  - nonnegative integers

Accompanying the set  $E$  of edges to the set  $V$  of vertices - we obtain the Hasse diagram where here down  $p, q, s \in N_0$ .

Namely

**Definition 2**

$$P = \langle V, E \rangle, V = \bigcup_{0 \leq p} \Phi_p, E = \{ \langle \langle j, p \rangle, \langle q, (p+1) \rangle \rangle \} \bigcup \{ \langle \langle 1, 0 \rangle, \langle 1, 1 \rangle \rangle \}, 1 \leq j \leq p_F, 1 \leq q \leq (p+1)_F.$$

**Definition 3** The prototype cobweb sub-poset is :  $P_m = \bigcup_{0 \leq s \leq m} \Phi_s$ .

In reference [2] a partially ordered infinite set  $P$  was introduced via descriptive picture of its Hasse diagram. Indeed, we may picture out the partially ordered infinite set  $P$  from the Definition 1 with help of the sub-poset  $P_m$  (rooted at  $F_0$  level of the poset) to be continued then ad infinitum in now obvious way as seen from the Fig.1 of  $P_5$  below. It looks like the Fibonacci rabbits' tree with a specific "cobweb".

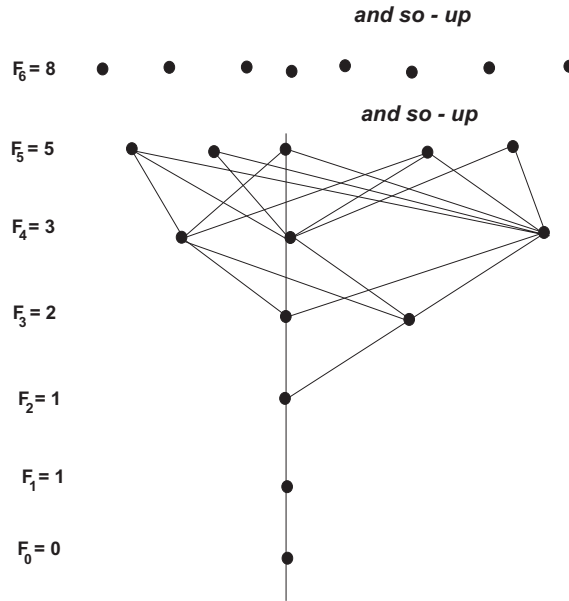


Fig. 1. Combinatorial interpretation of Fibonomial coefficients.

As seen above the *Fig.1.* displays the rule of the construction of the Fibonacci "cobweb" poset. It is being visualized clearly while defining this cobweb poset  $P$  with help of its incidence matrix. The incidence  $\zeta$  function [5,6,7] matrix representing uniquely just this cobweb poset  $P$  has the staircase structure correspondent with "cobwebbed" Fibonacci Tree i.e. a Hasse diagram of the particular partial order relation under consideration. This is seen below on the Fig.2.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{bmatrix}$$

Figure 2. The staircase structure of incidence matrix  $\zeta$

**Description** The "cob-viewer" encounters : in the zeroth - zero and in the  $k - th$  row (for  $k > 0$ )  $F_k - 1$  zeros right to the diagonal value 1 thus getting a picture of descending down to infinity led by diagonal direction with use of growing in size cobweb Fibonacci staircase tiled and build of the only up the diagonal zeros - note these are forbiddance zeros (they code no edge links along  $k - th$  levels ("generations") of the "cobwebbed" Fibonacci rabbits tree from [2].

This staircase structure of incidence [6,7] matrix  $\zeta$  which equivalently defines uniquely this particular cobweb poset was being recovered right from the Definition 1 and illustrative Hasse diagram in Fig.1 of Fibonacci cobweb poset. Let us say it again - if one decides to define the poset  $P$  by incidence matrix  $\zeta$  then must arrives at  $\zeta$  with this easily recognizable staircase-like structure of zeros in the upper part of this upper triangle incidence matrix  $\zeta$  just right from the picture (see [1] and [18] for recent references).

Let us recall [5,6,7] that  $\zeta$  is being defined for any poset as follows ( $p, q \in P$ ):

$$\zeta(p, q) = \begin{cases} 1 & \text{for } p \leq q, \\ 0 & \text{otherwise.} \end{cases}$$

The above  $\zeta$  characteristic matrix of the partial order relation in  $P$  has been expressed explicitly in [1] in terms of the infinite Kronecker delta matrix  $\delta$  from incidence algebra  $I(P)$  [5,6,7] as follows:

$$\zeta = \zeta_1 - \zeta_0$$

where for  $\langle x, y \rangle \in N_0 \times N_0$ ,

$$\zeta_1(x, y) = \sum_{k \geq 0} \delta(x + k, y)$$

while

$$\zeta_0(x, y) = \sum_{k \geq 0} \sum_{s \geq 1} \delta(x, F_{s+1} + k) \sum_{1 \leq r \leq (F_s - k - 1)} \delta(k + F_{s+1} + r, y).$$

Naturally

$$\delta(x, y) = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

**Important.** The knowledge of  $\zeta$  matrix explicit form enables one [6,7] to construct (count) via standard algorithms [6,7] the Möbius matrix  $\mu = \zeta^{-1}$  and other typical elements of incidence algebra perfectly suitable for calculating number of chains, of maximal chains etc. in finite sub-posets of  $P$ . Right from the definition of  $P$  via its Hasse diagram here now follow quite obvious observations .

**Observation 1**

*The number of maximal chains starting from The Root (level  $0_F$ ) to reach any point at the  $n - th$  level with  $n_F$  vertices is equal to  $n_F!$ .*

**Observation 2** ( $k > 0$ )

*The number of maximal chains **rooted in any fixed** vertex at the  $k - th$  level reaching the  $n - th$  level with  $n_F$  vertices is equal to  $n_F^m$ , where  $m + k = n$ .*

Indeed. Denote the number of ways to get along maximal chains from a **fixed point** in  $\Phi_k \rightarrow \Phi_n, n > k$  with the symbol

$$[fixed \Phi_k \rightarrow \Phi_n]$$

then obviously we have :

$$[\Phi_0 \rightarrow \Phi_n] = n_F!$$

and

$$[\Phi_0 \rightarrow \Phi_k] \times [fixed \Phi_k \rightarrow \Phi_n] = [\Phi_0 \rightarrow \Phi_n].$$

**Note** that the number  $[\Phi_k \rightarrow \Phi_n]$  of all maximal chains starting from the  $k - th$  level and ending at the  $n - th$  level equals to

$$[\Phi_k \rightarrow \Phi_n] = k_F \times n_F^m$$

, where  $m + k = n$ .

In order to find out the combinatorial interpretation of Fibonomial coefficients let us consider all such finite "max-disjoint" sub-posets rooted at the  $k - th$  level at any fixed vertex  $\langle r, k \rangle, 1 \leq r \leq k_F$  and ending at corresponding number of vertices at the  $n - th$  level ( $n = k + m$ ).

**Explanation:** "max-disjoint" means : sub-posets looked upon as families of **maximal** chains are disjoint copies . These copies are like  $P_m(k)_r$  copies isomorphic to  $P_m = P_m(0)_0$  - defined below for the sake of illustration.

In coordinate system this auxiliary illustrative cobweb sub-poset  $P_m(k)_r$  is defined as follows:

**Definition 4** Let  $\langle k, r \rangle \oplus \Pi$  denotes the shift of the set  $\Pi$  with the vector  $\langle k, r \rangle$  Let  $\Phi_o = \{\langle 0, 0 \rangle\}$ . Then we define:

$$P_m(k)_r = \langle V_m(k)_r, E_m(k)_r \rangle, V_m(k)_r = \langle k, r \rangle \oplus \bigcup_{0 \leq s \leq m} \Phi_s, \\ E_m(k)_r = \{ \langle (r+j), (k+s) \rangle, \langle (r+i), (k+s+1) \rangle \}, 0 \leq (r+j) \leq (k+s)_F, 1 \leq (r+i) \leq (k+s+1)_F \}.$$

**Observe**  $P_m(0)_0 = \langle V_m(0)_0, E_m(0)_0 \rangle \equiv \langle V_m, E_m \rangle \equiv P_m$ . Hence  $V_m(k)_r = \langle k, r \rangle \oplus V_m$ . Here, let us recall:  $P_m$  is the sub-poset of  $P$  rooted at the 0 - th level consisting of all intermediate level vertices up to  $m$  - th level ones - those from  $\Phi_m$  included (See : Fig.1.).

**A newly k-th level born sub-cob browsing.**

Consider now the following behavior of a sub-cob useful animal moving from any given point of the  $F_k$  "generation level" of the poset up and then up... It behaves as it has been born right there and can reach at first  $F_2$  vertices-points up, then  $F_3$  points up,  $F_4$  up... and so on - thus climbing up to the level  $F_{k+m} = F_n$  of the poset  $P$ . It can see and then potentially follow- one of its own thus accessible isomorphic *copy* of sub-poset  $P_m(k)_r$  in between the  $k$ -th and  $n$ -th levels. ( "It" behaves exactly as its Great Ancestor does born at the Source Root  $F_0$  - th level).

One of many of such "max-disjoint" copies isomorphic with the sub-poset  $P_m$ 's (- the copies rooted at any fixed point of the  $k$ -th level) might be then found as a choice to start maximal chains forwarding up to the  $n$  - th level - in the limits of the chosen copy.

**How many** *different* of such "max-disjoint" subposets *choices* can be made?

**Observation 3 ( $k > 1$ )**

Let  $n = k + m$ . The number max-disjoint sub-posets isomorphic to  $P_m$ , rooted at the  $k$  - th level and ending at the  $n$ -th level is equal to

$$(n - m)_F \times \frac{n_F^m}{m_F!} = (n - m)_F \times \binom{n}{m}_F \\ = k_F \times \binom{n}{k}_F = k_F \times \frac{n_F^k}{k_F!}.$$

Indeed. Consider the number of all max-disjoint isomorphic copies of  $P_m$  rooted at a fixed vertex  $\langle (r+j), k \rangle, 1 \leq (r+j) \leq k_F$ . Denote this number with the symbol

$$\binom{n}{k}_F$$

Recall that the number of all maximal chains from any point in  $\Phi_k$  to  $\Phi_n, n > k$  is equal to

$$[\Phi_k \rightarrow \Phi_n] = k_F \times n_F^m$$

Then one observes that :

$$(4) \quad k_F \times \binom{n}{k}_F \times [\Phi_0 \rightarrow \Phi_m] = [\Phi_k \rightarrow \Phi_n] = k_F \times n_F^m$$

where  $[\Phi_0 \rightarrow \Phi_m] = m_F!$  counts the number of maximal chains in any copy of the  $P_m$ . The factor  $k_F$  arises from symmetric input by the vertices of the  $k$  - th level.

For example: for the case  $k = 3$  and  $n = 4$ , according to the interpretation given above, there should be  $F_3 \times \binom{4}{1}_F = F_3 \times F_4! / F_1! F_3! = 6$  max-disjoint copies of  $P_1$  rooted at the third level and ending at the fourth level, which is exact.

For example: for the case  $k = 2$  and  $n = 4$ , according to the interpretation given above, there should be  $F_2 \times \binom{4}{2}_F = F_4!/F_2!F_2! = 6$  max-disjoint copies of  $P_2$  rooted at the second level - ending at the fourth level , which is exact.

For example: for the case  $k = 3$  and  $n = 5$ , according to the interpretation given above, there should be  $F_3 \times \binom{5}{3}_F = F_3 \times F_5!/F_3!F_2! = 30$  max-disjoint copies of  $P_2$  rooted at the third level and ending at the fifth level , which is exact.

For example: for the case  $k = 2$  and  $n = 5$ , according to the interpretation given above, there should be  $F_2 \times \binom{5}{3}_F = F_5!/F_3!F_2! = 15$  max-disjoint copies of  $P_3$  rooted at the second level and ending at the fifth level , which is exact.

For example: for the case  $k = 4$  and  $n = 5$ , according to the interpretation given above, there should be  $F_4 \times \binom{5}{1}_F = F_4F_5!/F_4!F_1! = 15$  the number of max-disjoint copies of  $P_1$  rooted at the fourth level and ending at the fifth level , which is exact.

**Important to Note** The reason for the restriction  $k > 1$  being applied is because  $F_1 = F_2$  and for  $k = 1$  Observation 3 is not true.

For example: for the case  $k=1$  and  $n = 4$ , according to the interpretation given above, there should be  $F_1 \times \binom{4}{1}_F = F_4!/F_1!F_3! = 3$  max-disjoint copies of  $P_3$  rooted at the thirist level and ending at the fourth level , which is not true.

For example: for the case  $k=1$  and  $n = 5$ , according to the interpretation given above, there should be  $F_1 \times \binom{5}{1}_F = F_5!/F_4!F_1! = 5$  max-disjoint copies of  $P_4$  rooted at the first level and ending at the fifth level which is not true.

### 3 Does Konvalina like interpretation of objects $F$ - selections from weighted boxed exist?

Binomial enumeration or finite operator calculus of Roman-Rota and Others is now the standard tool of combinatorial analysis. The corresponding  $q$ -binomial calculus ( $q$ -calculus - for short) is also the basis of much numerous applications (see [19,20] for altogether couple of thousands of respective references via enumeration and links). In this context Konvalina unified binomial coefficients look intriguing and much promising. The idea of  $F$ -binomial or Fibonomial finite operator calculus (see Example 2.1 in [17]) consists of specification of the general scheme - (see: [16,17] and references also to Ward, Steffensen ,Viskov , Markowsky and others - therein)- specification via the choice of the Fibonacci sequence to be sequence defining the generalized binomiality of polynomial bases involved (see Example 2.1 in [17]).Till now however we had been lacking alike combinatorial interpretation of Fibonomial coefficients. We hope that this note would help not only via Observations above but also due to coming next- observation where recurrence relation for Finonomial coefficients is is subjected to accordingly attempted combinatorial interpretation.

**Observation 4** ( $k > 0$ ) , (combinatorial interpretation of the recurrence)

The following known [11,14] recurrences hold

$$\binom{n+1}{k}_F = F_{k-1} \binom{n}{k}_F + F_{n-k+2} \binom{n}{k-1}_F$$

or equivalently

$$\binom{n+1}{k}_F = F_{k+1} \binom{n}{k}_F + F_{n-k} \binom{n}{k-1}_F$$

$$\binom{n}{0}_F = 1, \binom{0}{k}_F = 0,$$

$=k+m$ ). The first one for which

equals to  $F_{k+1}$  times number of different isomorphic copies of  $P_m$  - rooted at a fixed point on the  $k - th$  level (see Interpretation below) and

$$(5) \quad F_{n-k} \binom{n}{k-1}_F = F_{n-k} \binom{n}{n-k+1}_F$$

below.

[illegible]

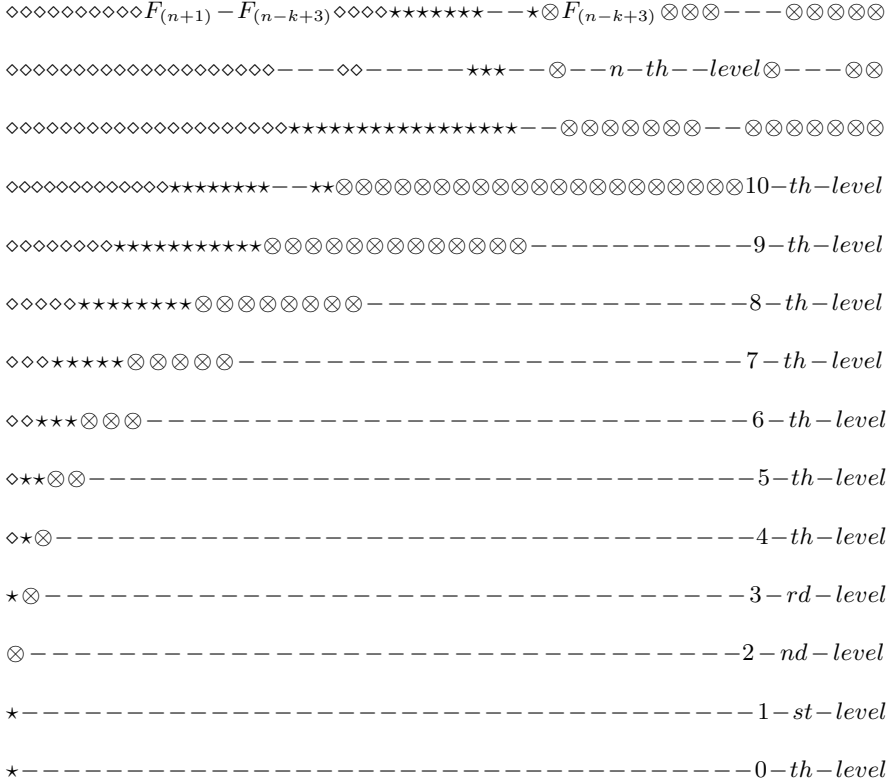
The Fig.3 illustrates how the two disjoint classes referred to in Observation 4 come into existence ( $r = 1, k = 4$ ). First: every cobweb sub-poset has the "trunk" of length

$\geq$  one (in the *Fig.3* it is the extreme left maximal chain). From any selected root-vertex in  $k - th$  level  $F_{(k+1)}$  trunks may be continued in  $F_{k+1}$  ways. A trunk of the  $P_{m+1}$  copy being chosen - for example the set of vertices  $\langle 1, s \rangle, k \leq s \leq (n+1)$  in the case of diamond cobweb poset selected in *Fig. 3* - the resulting sub-cobweb ends with  $F_{(n-k+2)}$  diamond vertices ("leafs") at  $(n+1) - th$  level. The different copies when shifted (in  $F_{k+1}$  ways - each ) up and correspondingly completed by - with the ultimate rightist maximal chain ending - the lacking part of now  $P_{m+1}$ 's copy - become now the different copies rooted at  $k - th$  level and ending at the  $(n+1)th$  level. This gives

$$F_{k+1} \left( \begin{matrix} n \\ k \end{matrix} \right)_F,$$

what constitutes the first summand of the corresponding recurrence.

Consider then the non- $\Phi_k$  level (then to be shifted) choice of the vertex .



**Figure 4. The non-diamond choice - two disjoint classes in  $P_{(n+1)}$ .**

The *Fig.4* continues to illustrate how the two disjoint classes referred to in *Observation 4* are introduced. Now - what we do we choose a vertex-root  $\otimes$  in  $(k-1) - th$  level in one of  $F_{(k-1)}$  ways. A trunk being chosen - say of the  $\otimes$  cobweb sub-poset in *Fig. 4* - it ends with  $F_{(n-k+2)}$   $\otimes$  vertices ("leafs") at  $n - th$  level. The number

of  $\otimes$  different isomorphic copies of  $P_m$  rooted at a fixed point at  $(k-1)$ -th level is equal to  $\binom{n}{k-1}_F$ . These copies become now copies rooted at  $k$ -th level while ending at the  $(n+1)$ -th level when shifted up the  $k$ -th and rooted at the same "diamond" root of the first choice being afterwards correspondingly completed by - with the ultimate leftist maximal chain ending -the lacking part of  $P_{m+1}$ 's copies with all other copies rooted there at  $k$ -th level and ending at the  $(n+1)$ -th level. The number of thus obtained different copies is equal to

$$F_{n-k} \binom{n}{k-1}_F.$$

All together this gives the number of all different cobweb sub-posets isomorphic copies ending at  $\Phi_{(n+1)}$  while starting from a fixed point of  $\Phi_k$  level. This number is equal to the sum of number from the two disjoint classes i.e.

$$\binom{n+1}{k}_F = F_{k+1} \binom{n}{k}_F + F_{n-k} \binom{n}{k-1}_F.$$

In this connection the intriguing question arises : May one extend-apply somehow Konvalina theorem [3,4] below so as to encompass also Fibonomial case under investigation ?

In [3,4] Konvalina considers  $n$  distinct boxes labeled with  $i \in [n]$ ,  $[n] \equiv \{1, \dots, n\}$  such that each of  $i$ -th box contains  $w_i$  distinct objects. John Konvalina uses the convention  $1 \leq w_1 \leq w_2 \leq \dots \leq w_n$ . Vector  $N^n \ni \vec{w} = (w_1, w_2, \dots, w_n)$  is the weigh vector then. Along with Konvalina considerations we have from [3]:

#### The Konvalina Theorem 1

Let  $\vec{w} = (w_1, w_2, \dots, w_n)$  where  $1 \leq w_1 \leq w_2 \leq \dots \leq w_n$ . Then

I.

$$C_k^n(\vec{w}) = C_k^{n-1}(\vec{w}) + w_n C_{k-1}^{n-1}(\vec{w})$$

II.

$$S_k^n(\vec{w}) = S_k^{n-1}(\vec{w}) + w_n S_{k-1}^{n-1}(\vec{w}).$$

Here  $C_k^n(\vec{w})$  denotes the generalized binomial coefficient of the first kind with weight  $\vec{w}$  and it is the number of ways to select  $k$  objects from  $k$  (necessarily distinct !) of the  $n$  boxes with constrains as follows : choose  $k$  distinct labeled boxes

$$i_1 < i_2 < \dots < i_k$$

and then choose one object from each of the  $k$  distinct boxes selected. Naturally one then has [3]

$$C_k^n(\vec{w}) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} w_{i_1} w_{i_2} \dots w_{i_k}.$$

Complementarily  $S_k^n(\vec{w})$  denotes the generalized binomial coefficient of the second kind with weight  $\vec{w}$  and it is the number of ways to select  $k$  objects from  $k$  (not necessarily distinct) of the  $n$  boxes with constrains as follows [3]: choose  $k$  not necessarily distinct labeled boxes

$$i_1 \leq i_2 \leq \dots \leq i_k$$

and then choose one object from each of the  $k$  (not necessarily distinct) boxes selected. Obviously one then has

$$S_k^n(\vec{w}) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} w_{i_1} w_{i_2} \dots w_{i_k}.$$

here the natural question arises : how are we to extend Konvalina theorem [3,4] above so as to encompass also Fibonomial case under investigation ?

**Information I** : about the preprint [21] entitled *Determinants, Paths, and Plane Partitions* by Ira M. Gessel, X. G. Viennot [21].

Right after Theorem 25 - Section 10 , page 24 in [21])- relating the number  $N(R)$  of nonintersecting  $k$ -paths to Fibonomial coefficients via  $q$ -weighted type counting formula- the authors express their wish worthy to be quoted: "*it would be nice to have a more natural interpretation then the one we have given*"... " *R. Stanley has asked if there is a binomial poset associated with the Fibonomial coefficients...*" - Well. The cobweb locally finite infinite poset by Kwasniewski from [15,18,1,2] is not of binomial type. Recent incidence algebra origin arguments [22] seem to make us not to expect binomial type poset come into the game. The  $q$ -weighted type counting formula from [21] gives rise to an interesting definition of Fibonomial coefficients all together with its interpretation in terms of nonintersecting  $k$ -paths due to the properties of binomial determinant. Namely ,following [21] let us consider points  $P_i = \langle 0, -i \rangle$  and  $Q_i = \langle n + i, -n + i \rangle$ . Let  $R = \{r_1 < r_2 < \dots < r_k\} \equiv R(\vec{r})$  be a subset of  $\{0, 1, \dots, n\} \equiv [n + 1]$ . Let  $N(R)$  denotes the number of non-intersecting  $k$ -paths from  $\langle P_{r_1}, \dots, P_{r_k} \rangle$  to  $\langle Q_{r_1}, \dots, Q_{r_k} \rangle$ . Then  $\det \begin{pmatrix} r_i & \\ & n - r_{k+1-j} \end{pmatrix} = N(R)$ . The  $q$ -weighted type counting formula from [21] then for  $q = 1$  means that

$$\binom{n+1}{k}_F = \sum_{R(\vec{r})} N(R).$$

In view of [21] another question arises - what is the relation like between these two: Gessel and Viennot [21] non-intersecting  $k$ -paths and cobweb sub-poset [18,2,3] points of view?

**Information II** : on the partial ordered poset and Fibonacci numbers paper [23] by Istvan Beck. The author of [23] shows that  $F_n$  equals to the number of of ideals in a simple poset called "fence" . This allows Him to infer via combinatorial reasoning the identities :

$$\begin{aligned} F(n) &= F(k)F(n+1-k) + F(k-1)F(n-k) \\ F(n) &= F(k-1)F(n+1-(k-1)) + F(k-2)F(n-(k-1)). \end{aligned}$$

A straightforward application of these above is the confirmation - just by checking - the intriguing validity of recurrence relation for Fibonomial coefficients . As we perhaps might learn from this note coming to the end - both the Fibonomial coefficients as well as their recurrence relation are interpretable along the classical historically established manner referring to the number of objects' choices - this time these are partially ordered sub-sets here called the cobweb sub-posets - the effect of the diligent spider's spinning of the maximal chains cobweb during the arduous day spent on the infinite Fibonacci rabbits' growth tree.

**Historical Memoir Remark** The Fibonacci sequence origin is attributed and referred to the first edition (lost) of "Liber abaci" (1202) by Leonardo Fibonacci [Pisano] (see second edition from 1228 reproduced as *Il Liber Abaci di Leonardo Pisano* pubblicato secondo la lezione Codice Magliabeciano by Baldassarre Boncompagni in *Scritti di Leonardo Pisano* vol. 1, (1857) Rome).

**Historical Quotation Remark** As accurately noticed by Knuth and Wilf in [14] the recurrent relations for Fibonomial coefficients appeared already in 1878 Lukas work [11]. In our opinion - Lucas's *Théorie des fonctions numériques simplement périodiques* is the far more non-accidental context for binomial and binomial-type coefficients - Fibonomial coefficients included.

While studying this mentioned important and inspiring paper by Knuth and Wilf [14] and in the connection with a context of this note a question raised by the authors with respect to their formula (15) is worthy to be repeated : *Is there a "natural" interpretation....* - May be then fences from [23] or cobweb posets or ... "Natural" naturally might have many effective faces ...

**Acknowledgements** I am very much indebted to Mgr Ewa Krot - for her substantial critical remarks allowing to present my exposition in hopefully better shape.

## References

- [1] A. K. Kwaśniewski, *The logarithmic Fib-binomial formula* Advanced Stud. Contemp. Math. **9** No 1 (2004):19-26
- [2] A. K. Kwasniewski *Information on combinatorial interpretation of Fibonomial coefficients* Bull. Soc. Sci. Lett. Lodz Ser. Rech. Deform. 53, Ser. Rech.Deform. **42** (2003): 39-41 ArXiv: math.CO/0402291 v1 18 Feb 2004
- [3] J. Konvalina , *Generalized binomial coefficients and the subset-subspace problem* , Adv. in Appl. Math. **21** (1998) : 228-240
- [4] J. Konvalina , *A Unified Interpretation of the Binomial Coefficients, the Stirling Numbers and the Gaussian Coefficients* The American Mathematical Monthly **107**(2000):901-910
- [5] Gian-Carlo Rota "On the Foundations of Combinatorial Theory, I. Theory of Mbius Functions"; Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol.2 , 1964 , pp.340-368.
- [6] E. Spiegel, Ch. J. O'Donnell *Incidence algebras* Marcel Dekker, Inc. Basel 1997 .
- [7] Richard P. Stanley *Enumerative Combinatorics I*, Wadsworth and Brooks Cole Advanced Books and Software, Monterey California, 1986
- [8] Goldman J. Rota G-C. *The Number of Subspaces in a vector space* in Recent Progress in Combinatorics (W. Tutte, Ed.): 75-83, Academic Press, San Diego, 1969, see: ( J. Kung, Ed.) "Gian Carlo Rota on Combinatorics" Birkhuser, Boston (1995):217-225
- [9] Goldman J. Rota G-C. *On the Foundations of Combinatorial Theory IV; finite-vector spaces and Eulerian generating functions* Studies in Appl. Math. **49** (1970): 239-258
- [10] J. Kung *The subset-subspace analogy* ( J. Kung, Ed.) "Gian Carlo Rota on Combinatorics" Birkhuser, Boston (1995):277-283
- [11] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, American Journal of Mathematics **1** (1878): 184-240; (Translated from the French by Sidney Kravitz), Ed. D. Lind, Fibonacci Association, 1969.
- [12] G. Fontené, *Généralisation d'une formule connue*, Nouvelles Annales de Mathématiques (4) **15** (1915), 112.
- [13] H. W. Gould, *The bracket function and Fontené-Ward generalized binomial coefficients with applications to Fibonomial coefficients*, The Fibonacci Quarterly **7** (1969), 23-40.
- [14] D. E. Knuth, H. S. Wilf *The Power of a Prime that Divides a Generalized Binomial Coefficient* J. Reine Angev. Math. **396** (1989): 212-219
- [15] A. K. Kwaśniewski, *More on Combinatorial interpretation of Fibonomial coefficients*, Inst. Comp. Sci. UwB/Preprint no. 56, November 2003.
- [16] A. K. Kwaśniewski, *Towards  $\psi$ -extension of finite operator calculus of Rota*, Rep. Math. Phys. **47** no. 4 (2001), 305-342. ArXiv: math.CO/0402078 2004

- [17] A. K. Kwaśniewski, *On simple characterizations of Sheffer  $\Psi$ -polynomials and related propositions of the calculus of sequences*, Bull. Soc. Sci. Lettres Łódź **52**, Sér. Rech. Déform. **36** (2002), 45–65. ArXiv: math.CO/0312397 2003
- [18] A. K. Kwaśniewski, *Comments on combinatorial interpretation of fibonomial coefficients - an email style letter*, Bulletin of the Institute of Combinatorics and its Applications, **42** September (2004): 10-11
- [19] T. Ernst, *The History of  $q$ -Calculus and a new Method*, <http://www.math.uu.se/~thomas/Lics.pdf> 19 December (2001), (Licentiate Thesis). U. U. D. M. Report (2000).
- [20] A. K. Kwaśniewski, *First Contact Remarks on Umbra Difference Calculus References Streams* Inst. Comp. Sci. UwB Preprint No **63** (January 2004)
- [21] Ira M. Gessel, X. G. Viennot *Determinant Paths and Plane Partitions* preprint (1992) <http://citeseer.nj.nec.com/gessel89determinants.html>
- [22] A.K.Kwasniewski *The second part of on duality triads' paper-On fibonomial and other triangles versus duality triads* Bull. Soc. Sci. Lett. Lodz Ser. Rech. Deform. **53**, Ser. Rech. Deform. **42** (2003): 27 -37 ArXiv: math.GM/0402288 v1 18 Feb. 2004
- [23] I. Beck *Partial Orders and the Fibonacci Numbers* The Fibonacci Quarterly **26** (1990): 172-174 .